Descriptive Set Theory HW 1

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Problem 1. Let (X, τ) be a second-countable topological space.

1. Show that X has at most continuum many open subsets.

2. Prove that any strictly monotone sequence $(U_\alpha)_{\alpha<\gamma}$ of open subsets of X has countable length.

3. Show that every monotone sequence $(U_\alpha)_{\alpha<\omega_1}$ of opens sets eventually stabilizes.

4. Conclude that the above holds for closed sets as well.

Solution.

1. Fix a countable basis $(B_n)_{n<\omega}$ for X. Notice that the open subsets of X are precisely unions of elements from this countable basis. With this in mind, define $f : \tau \to \wp(\omega)$ by sending $U \in \tau$ to the set $\{n \colon B_n \subseteq U\}$. Notice that $f(U)$ is maximal such that $U = \bigcup_{n\in f(U)} B_n$. It's straightforward to check that $U \subseteq V$ iff $f(U) \subseteq f(V)$, and so it follows that f is injective, yielding the result. Because the map $U \mapsto X - U$ is a bijection between the open and closed sets, the result also holds for closed sets.

2. Let f be the function defined before. It follows from above that $U \subsetneq V$ iff $f(U) \subsetneq f(V)$. The key fact is that there cannot be an uncountable strictly increasing or decreasing sequence of subsets of ω, as ω is countable. To see this, if we had for example that $(A_\alpha)_{\alpha<\omega_1}$ was a strictly decreasing sequence of subsets of ω, we could choose for each $\alpha < \omega_1$ an element $n_\alpha \in A_\alpha - A_{\alpha+1}$, which defines an injection from ω_1 into ω.

   So, using the observation above, any uncountable strictly monotone sequence $(U_\alpha)_{\alpha<\gamma}$ of open sets would correspond either to the uncountable strictly increasing or decreasing sequence $(f(U_\alpha))_{\alpha<\gamma}$ of subsets of ω, contradicting the above remark. The result for closed subsets of X follows by taking complements.

3. Assume instead there was a monotone sequence $(U_\alpha)_{\alpha<\omega_1}$ of opens sets that never stabilized. Without loss of generality, assume that it were increasing. Then for each $\alpha < \omega_1$, there would be a $\beta > \alpha$ such that $U_\alpha \subsetneq U_\beta$. Then, the set $\{\alpha < \omega_1 \colon (\forall \beta < \alpha)\ U_\beta \subsetneq U_\alpha\}$ is an unbounded (and hence uncountable) subset of ω_1. This induces an uncountable strictly increasing sequence of open sets, contradicting (2). The result for closed sets follows by taking complements.

4. Remarks were made above concerning this question.

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Problem 2. Prove that any separable metric space has cardinality at most continuum. Counterexample for general separable Hausdorff spaces?

Solution. Let (X, d) be such a metric space, and $D \subseteq X$ witness separability. Define a map $f \colon X \to \mathbb{R}^D$ by $f(x)(a) = d(x, a)$. The point is that elements of X are determined by their distances from elements in a dense subset. This map is an injection, as if $x \neq y$, there's an $r > 0$ such that $d(x, y) \geq r$. By density, there's an $a \in D$ such that $d(x, a) < \frac{r}{2}$. Triangle inequality implies that $d(y, a) \geq \frac{r}{2}$, yielding that $f(x)(a) \neq f(y)(a)$. Since D is countable, $\mathbb{R}^D$ has cardinality continuum, and the result follows.

For the counterexample, it is known that the product of continuum many separable Hausdorff spaces is separable, and so $2^{\mathbb{R}}$ is a separable Hausdorff space. ⋆

Problem 3. Let (X, d) be a metric space.

1. Show that X is complete iff every decreasing sequence of nonempty closed sets $(B_n)_{n<\omega}$ with $\operatorname{diam}(B_n) \to 0$ has nonempty intersection (is a singleton).

2. Show that $\operatorname{diam}(B_n) \to 0$ cannot be dropped.

Solution.

1. Let X be complete and $(B_n)_{n<\omega}$ a decreasing sequence of closed sets with $\operatorname{diam}(B_n) \to 0$. Then consider the sequence (x_i) by choosing $x_i \in B_i$. This sequence is Cauchy as given any $\varepsilon > 0$, there's some N such that $k \geq N$ implies $\operatorname{diam}(B_k) < \varepsilon$. For any two $n, m \geq N$, since the sequence is decreasing, $x_n, x_m \in B_k$, implying that $d(x_n, x_m) < \varepsilon$. By completeness, this sequence has a limit x. Then, $x \in \bigcap_{n<\omega} B_n$ as each

is closed and the sequence (x_i) is eventually in any B_n. To see that this intersection is a singleton, observe that if $y \in \bigcap_{n<\omega} B_n$, then, for any $\varepsilon > 0$, $d(x,y) < \varepsilon$ as $\operatorname{diam}(B_n) \to 0$. This implies that $x = y$.

Going in the other direction, fix a Cauchy sequence (x_n) in X. Use Cauchy-ness to inductively construct an strictly increasing sequence (N_k) such that, for $k > 0$, $d(x_n, x_m) < 2^{-(k+1)}$ when $n, m \geq N_k$. Then let $B_k = B[x_{N_k}, 2^{-k}]$ be the closed ball of radius 2^{-k} around x_{N_k}. It's clear that $\operatorname{diam}(B_k) \to 0$.

To show this sequence is decreasing, fix $m \geq k+1$. If $y \in B[x_{N_m}, 2^{-m}]$, then by construction, $N_m > N_k$, and so $d(x_{N_m}, x_{N_k}) < 2^{-(k+1)}$. Since $2^{-m} \leq 2^{-(k+1)}$, it follows

$$d(y, x_{N_k}) \leq d(y, x_{N_m}) + d(x_{N_m}, x_{N_k}) < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k},$$

yielding $y \in B[x_{N_k}, 2^{-k}]$. By assumption, there's a unique $x \in \bigcap_{n<\omega} B_n$. It's easy to show $(x_n) \to x$. It follows that X is complete by definition.

2. To see that $\operatorname{diam}(B_n) \to 0$ cannot be dropped, consider the Polish space ω with the discrete metric. Every set is closed, and so $[n, \omega)_{n<\omega}$ forms a decreasing sequence of closed sets with empty intersection. With the discrete metric, each such set has diameter 1.

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Problem 4. Show G_δ sets are closed under finite unions. Equivalently, F_σ sets are under finite intersections.

Solution. Let $A = \bigcap_{i<\omega} U_i$ and $B = \bigcap_{i<\omega} V_i$ be G_δ sets. The F_σ case follows by taking complements. The key fact is that for sets X and $\{Y_i\}_{i\in I}$, we have $X \cup \bigcap_i Y_i = \bigcap_i (Y_i \cup X)$. In particular,

$$A \cup B = \bigcap_{n<\omega} (V_n \cup A) = \bigcap_{n<\omega} (\bigcap_{m<\omega} (V_n \cup U_m)) = \bigcap_{n,m\in\omega} (V_n \cup U_m).$$

The final intersection is countable and $V_n \cup U_m$ is open for any $n, m < \omega$, so $A \cup B$ is G_δ by definition. ⋆

Problem 5. Show the following:

1. Baire space is homeomorphic to a G_δ subset of the Cantor space.
2. The irrationals is homeomorphic to the Baire space.

Solution.

1. It's enough show to that Baire space is homeomorphic to a subspace of Cantor space, as a Polish subspace of a Polish space is G_δ. Define the map $\varphi\colon \omega^\omega \to 2^\omega$ by $\varphi(x) = 0^{x_0}10^{x_1}10^{x_2}1\ldots$, where 0^{x_i} is 0 repeated x_i-many times. Observe that φ injects ω^ω onto the set of all binary sequences with 1 appearing infinitely often. Now, φ is a homeomorphism as for any $x \in \omega^\omega$, the first n digits of $\varphi(x)$ are determined by (at most) the first n digits of x. Similarly, φ^{-1} is continuous as the first n digits of x are determined by the first $\sum_{i<n} x_i + n$ digits of $\varphi(x)$.

2. Let $\mathbb{D}$ denote the space ω^ω where we replace our first copy of ω with $\mathbb{Z}$ endowed with the discrete topology, and each subsequent factor of ω doesn't contain 0. $\mathbb{D}$ is homeomorphic to ω^ω, so it's enough to show that the irrationals is homeomorphic to $\mathbb{D}$. Following the hint, given an irrational number x, we can write it uniquely as follows:

$$x = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{x_4 + \ddots}}}} \tag{1}$$

   where x_0 is an integer and $x_i < \omega$ for each $i > 0$. This is apparently denoted $[x_0; x_1, x_2 \ldots]$. Further, each such continued fraction like the RHS of (1) is an irrational number. Define $\varphi\colon \mathbb{D} \to \mathbb{I}$ by $x \mapsto [x_0; x_1, x_2 \ldots]$. Then φ is a homeomorphism. Indeed, φ is continuous as given an irrational $\varphi(x) = [x_0; x_1, x_2 \ldots]$ and any ε-ball B around $\varphi(x)$, membership in B will depend on only the first n many digits of x (for some sufficiently large n). Going in the other direction, given $x \in \mathbb{D}$ and the first n digits of x, we can find an ε-ball around $[x_0; x_1, x_2 \ldots]$ such that any $y \in \mathbb{D}$ with $\varphi(y)$ inside this ball will agree with x on the first n digits.

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Problem 6. Let $T \subseteq A^{<\mathbb{N}}$ be a tree and suppose it is finitely branching. Prove that $[T]$ is compact.

Solution. Using the notation from Anush's notes, let $T(\sigma) = \{x \in A \colon \sigma^\frown x \in T\}$ for σ a finite sequence from A. Without loss of generality, assume that T is pruned and $[T]$ is nonempty. Now, let $(x_n)_{n<\omega}$ be a sequence of elements of $[T]$. It's enough to find a convergence subsequence. Construct such a subsequence $(x_{n_i})_{i<\omega}$ and sequence $(a_i)_{i<\omega}$ by induction such that at each stage k, the set $(x_n)_{n<\omega} \cap N(x_{n_k} \restriction k)$ is infinite, and that any two elements in this subsequence chosen after stage k will have $\langle a_0, \ldots, a_k \rangle$ as its first $k+1$ digits.

We do this in the following way: given x_{n_k}, $T(x_{n_k} \restriction k)$ is finite as our tree T is finitely branching. By the induction hypothesis there's an $a_k \in A$ such that the basic open set $N((x_{n_k} \restriction k)^\frown a_k)$ contains infinitely many elements from $(x_n)_{n<\omega}$. Choose $x_{n_{k+1}}$ to be such an element in $N((x_{n_k} \restriction k)^\frown a_k)$. In particular, notice that $x_{n_{k+1}} \restriction k = x_{n_k} \restriction k$. This completes the inductive definition.

Finally, define $x \in [T]$ by $x(k) = a_k$. We claim that $(x_{n_i}) \to x$. Indeed, notice that for any length m, we have by construction that any sequence element x_{n_i} chosen after stage m will extend $\langle a_0, \ldots, a_{m-1} \rangle$, and therefore agree with x on its first m digits. It follows that $(x_{n_i}) \to x$ as desired, and therefore that $[T]$ is compact. ⋆

Problem 7. Let S, T be trees on set A, B. Show that for any continuous function $f \colon [S] \to [T]$, there's a monotone map $\varphi \colon S \to T$ such that $f = \varphi^*$.

Solution. In what follows let N_s be shorthand for $N_s \cap [S]$ when s is a finite sequence from S (and similarly for N_t). Without loss of generality, assume that both $[S]$ and $[T]$ are nonempty. Following the hint, define $\varphi(s)$ to be the longest $t \in T$ such that $|t| \leq |s|$ and $N_t \supseteq f(N_s)$. The only hiccup may be if $N_s = \varnothing$; i.e., nothing in $[S]$ extends s. In this case we may set $\varphi(s)$ to be $\varphi(s')$, where $s' \subseteq s$ is the largest such that $N_{s'}$ isn't empty. Such a s' always exists. We need to check that φ is well defined. To see this, observe that if $|t_0| \leq |t_1|$, they satisfy $f(N_s) \subseteq N_{t_0}$ and $f(N_s) \subseteq N_{t_1}$, and N_s is nonempty, then $t_0 \subseteq t_1$.

To see what φ is doing, given $x \supseteq s$, we compute all the corresponding $f(x)$, and set $\varphi(s)$ as the largest initial segment (of length at most $|s|$) shared by all $f(x)$. With this in mind, it's not difficult to see that $\varphi(x)$ is monotone. Indeed, if $s \subseteq s'$, then when we compute $f(x)$ for extensions x of s', they all must at least extend $\varphi(s)$, as each x extends s as well. Since $\varphi(s')$ is the longest such element of T of length at most $|s'|$, it follows that $\varphi(s) \subseteq \varphi(s')$.

Finally, we check that $f = \varphi^*$. Recall, $\varphi^*(x) = \bigcup_{n<\omega} \varphi(x|n)$, where $\mathrm{dom}(\varphi^*)$ is all $x \in [S]$ such that $\lim_n |\varphi(x|n)| = \infty$. We first show that $[S] = \mathrm{dom}(\varphi)$. Given $x \in [S]$ and $k < \omega$, since f is continuous, we can find some $m \geq k$ such that $f(N_{x|m}) \subseteq N_{f(x)|k}$. We then have by definition that $k = |f(x) \restriction k| \leq |\varphi(x|m)|$. But, then $\lim_n |\varphi(x|n)| = \infty$, as φ is monotone and k was arbitrary.

The result follows after checking that $f(x) = \varphi^*(x)$. To see this, given k, we perform the argument in the above paragraph to find an $m \geq k$ such that $f(N_{x|m}) \subseteq N_{f(x)|k}$. Now, by definition, $f(N_{x|m}) \subseteq N_{\varphi(x|m)}$. Like above, we know that $|f(x) \restriction k| \leq |\varphi(x|m)|$, and so the observation from the first paragraph to check that φ was well-defined implies that in fact $f(x)|k \subseteq \varphi(x|m)$. As $f(x)$ and $\varphi^*(x)$ are both sequences, it finally follows that $f(x) = \bigcup_n f(x)|n = \bigcup_m \varphi(x|m) = \varphi^*(x)$. ⋆

Problem 8. Let (X, d) be a metric space. Show the following are equivalent:

1. X is compact.
2. Every sequence in X has a convergent subsequence.
3. X is complete and totally bounded.

Solution. We proceed by following Anush's outline.

(1) ⇒ (2): Given a sequence (x_n), let K_m be the closure of $\{x_n\}_{m\geq n}$. The collection of all such K_m have the finite intersection property, as they're the closure of the tails of the given sequence. Since X is compact, $\bigcap_m K_m$ is nonempty. Fix $x \in \bigcap_m K_m$. We now construct a subsequence $(x_{n_i})_i$ that converges to x: First, set $x_{n_0} = x_0$. Now, given x_{n_i}, we know by definition of closure that $B(x, 1/(n_i+1))$ intersects K_{n_i+1}. So, fix $m > n_i$ such that $x_m \in B(x, 1/(n_i+1))$. Set $x_{n_{i+1}} = x_m$. By construction, $n_i < n_{i+1}$ for all i, and $d(x, x_{n_{i+1}}) < \frac{1}{n_i+1} \to 0$ as $i \to 0$. It's straightforward to check that $(x_{n_i})_i \to x$.

(2) ⇒ (3): If X isn't complete, then there's a Cauchy sequence without a convergent subsequence. But this implies that the Cauchy sequence cannot converge. For total boundedness, assume instead that there's an $\varepsilon > 0$ such that X cannot be covered by finitely many ball of radius ε. We construct a sequence $(x_n)_{n<\omega}$ without a convergent subsequence. Since X isn't totally bounded, it must be non empty, so fix $x_0 \in X$. Given $(x_i)_{i<n}$, we have by hypothesis that $\bigcup_{i<n} B(x_i, \varepsilon) \neq X$. So, fix $x_{i+1} \notin \bigcup_{i<n} B(x_i, \varepsilon)$. By construction, we have that $d(x_i, x_j) \geq \varepsilon$ for all $i, j < \omega$, so certainly no subsequence of $(x_n)_{n<\omega}$ could converge.

(3) ⇒ (2): Fix a sequence $(x_n)_{n<\omega}$. If $x_i = x_j$ occurs for infinitely many pairs i, j, then we're done. Otherwise, by thinning to a subsequence we may

assume that $(x_n)_{n<\omega}$ is injective. Next, given $a \in X$ and $\varepsilon > 0$, let's write $S_{a,\varepsilon} = \{x_i \colon x_i \in B(a,\varepsilon)\}$. We construct a subsequence $(x_{n_i})_{i<\omega}$ and sequence $(a_i)_{i<\omega}$ as follows:

For the base case, first observe there's a 2^0-net F. By the pigeonhole principle and since our sequence is injective, there's an $a_0 \in F$ such that $S_{a_0,1}$ is infinite. Choose $x_{n_0} \in S_{a_0,1}$. Next, given x_{n_i} and a_i for $i \leq k$, we know that there's a finite $2^{-(k+1)}$-net F. By the pigeonhole principle and induction, there's an $a_{k+1} \in F$ such that $\bigcap_{i\leq k} S_{a_i,2^{-i}} \cap S_{a_{k+1},2^{-k-1}}$ is infinite. In particular, we may fix $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in \bigcap_{i\leq k} S_{a_i,2^{-i}} \cap S_{a_{k+1},2^{-k-1}}$.

By construction it follows that $(x_{n_i})_{i<\omega}$ is Cauchy (and thus converges by the previous paragraph). To see this, for any $\varepsilon > 0$, fix k such that $2^{-k-1} < \varepsilon$. Then, for all $i, j \geq k+1$, $x_{n_i}, x_{n_j} \in S_{a_{k+1},2^{-k-1}}$ by construction. It then follows that

$$d(x_{n_i}, x_{n_j}) \leq d(x_{n_i}, a_{k+1}) + d(x_{n_j}, a_{k+1}) < 2^{-k-1} + 2^{-k-1} = 2^{-k},$$

and we win.

(2) and (3) $\Rightarrow$ (1): Doing what Anush told me to do, I looked up Thm 0.25 in Folland. The main idea is that, given an open subcover $\{U_\alpha\}$, we find an $\varepsilon > 0$ such that every ε-ball B is contained in a U_{α_B}. For then we win because then we could cover X by a finite number of ε-balls B_i, implying $\{U_{\alpha_{B,i}}\}_i$ is a finite subcover. $\star$